

# Quantum fluctuations of coupled dark solitons in trapped Bose-Einstein condensates

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We show that the quantum fluctuations associated with the Bogoliubov quasiparticle vacuum can be strongly concentrated inside dark solitons in a trapped Bose Einstein condensate. We identify a finite number of anomalous modes that are responsible for such quantum phenomena. The fluctuations in these anomalous modes correspond to the ‘zero-point’ oscillations in coupled dark solitons.

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Dark solitons in quasi one-dimensional Bose-Einstein condensates (BEC) are coherent structures created from macroscopic excitations. Recent experiments have realized such dark solitons by using the phase imprinting technique [1,2], and several theoretical studies have analyzed the stability and dynamical properties [3–7], as well as novel soliton-pair solutions in two-component Bose condensates [8,9]. Like other familiar topological states in BEC, dark solitons are described by macroscopic wavefunctions. Even though atoms are excited, most of them still share the same wavefunction. Such a quantum degeneracy is the basis of treating the many-particle system as a single coherent matter wave. This raises a fundamental issue about the robustness of quantum degeneracy as a BEC is excited to higher energy states that typically contain more than one solitons. The presence of multiple dark solitons and their mutual interaction would introduce extra quantum fluctuations affecting the macroscopic coherence. In this paper we investigate the mechanism of such quantum decoherence effects.

We approach the problem by studying the quantum fluctuations associated with a chain of dark solitons in a trapped condensate. Specifically, we are interested in the minimal fluctuations represented by the *quasiparticle vacuum* in the Bogoliubov theory. In the case of ground state condensates, the corresponding quasiparticle vacuum leads to a cloud of incoherent atoms (usually called condensate depletion) not described by the Gross-Pitaevskii mean field equation [10]. Such incoherent atoms are typically a small fraction of the condensate, their effects can be ignored in most situations involving dilute condensates. However, we find that the quasiparticle vacuum associated with the dark solitons has distinctive features accessible for experimental observations. In this paper we show that incoherent atoms can be strongly

concentrated at the notches, and in addition we discover that this phenomenon is mainly the quantum fluctuations in a finite number of collective modes. The particle-like behavior of dark solitons is the key to interpret our findings. We shall see how dark solitons under confinement couple together as particles to execute collective oscillations. Quantum uncertainties in both the positions and momenta impose a fundamental limit of the degree of ‘darkness’ inside the dark solitons.

To begin we consider a Bose-Einstein condensate of weakly interacting atoms confined in a harmonic potential. A quasi one-dimensional condensate can be achieved in a trap in which the transverse motion is tightly confined (frozen) and only the longitudinal motion remains active. Given that  $m$  is the atomic mass and  $\omega_T$  is the trap frequency, the second quantized system is modeled by the Hamiltonian (in units of  $\hbar\omega_T$ ),

$$H = \frac{1}{2} \int dx \hat{\Psi}^\dagger \left( -\frac{\partial^2}{\partial x^2} + x^2 + \bar{\eta} \hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi}. \quad (1)$$

Here the spatial coordinate  $x$  is in units of  $(\hbar/m\omega_T)^{1/2}$ , and  $\bar{\eta}$  is the interaction strength between atoms averaged over the transverse area [11]. As usual, the field operator  $\hat{\Psi}(x, t)$  can be decomposed into a dominant coherent part and a fluctuation part, i.e.,  $\hat{\Psi} = \sqrt{N} \Phi + \hat{\phi}$ , where  $N$  is the number of condensate atoms. The mean field wavefunction  $\Phi = \Phi_n e^{-i\mu_n t}$  is the  $n$ -th excited state of the BEC satisfying the nonlinear Schrödinger eigen-equation,

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \Phi_n + \frac{1}{2} x^2 \Phi_n + g |\Phi_n|^2 \Phi_n = \mu_n \Phi_n \quad (2)$$

where  $\mu_n$  is the eigen-energy and  $g \equiv N\bar{\eta}$ . The quantum number  $n$  is the number of nodes appearing in  $\Phi_n$ . Therefore  $\Phi_n$  represents a static  $n$ -dark soliton solution.

In Fig. 1, we show the typical shapes of  $\Phi_n$  obtained from exact numerical solutions for the case  $g = 200$ . Although there are no simple analytic expressions for  $\Phi_n$ , we find that variational methods are useful to approximate these wavefunctions. We construct the trial wavefunction as a product of single dark soliton solutions embedded in a smooth background with the ground state profile, i.e.,

$$\Phi_n(x) \approx \alpha_n \Phi_0(x) \prod_{j=1}^n \tanh \left( \frac{x - q_j}{l_0} \right). \quad (3)$$

Here  $\alpha_n$  is a normalization constant,  $q_j$  is the position of the  $j$ -th dark soliton, and  $l_0^{-1} \equiv g^{1/2} \Phi_0(0)$  is the

inverse of the healing length (neglecting the slight position dependence near the trap center). The ground state wavefunction  $\Phi_0(x)$  can be further approximated by  $g^{-1/2}(\mu_0 - x^2/2)^{1/2}$  in the Thomas-Fermi limit  $g \gg 1$ . We treat  $q_j$  as variation parameters such that the mean field energy

$$E_n = \frac{N}{2} \int dx \left( \left| \frac{\partial \Phi_n}{\partial x} \right|^2 + x^2 |\Phi_n|^2 + g |\Phi_n|^4 \right) \quad (4)$$

is stationary. With the trial wavefunction given in Eq. (3), the  $q_j$  obtained have about 1% error compared with the exact numerical results in Fig. 2. Such a good agreement is understood from the fact that dark solitons are sufficiently separated from each other by several healing lengths  $l_0$ , which allows the product form in (3) to be a close approximation in the Thomas-Fermi limit. Indeed, we find that the separation between the solitons is between  $3l_0$  to  $6l_0$  for a wide range of interaction strengths  $10^2 < g < 10^3$ . We remark that dark solitons repel each other [12]. The mechanical equilibrium is attained by the inhomogeneous condensate background that provides a confinement ‘force’ holding the solitons together [5].

Having described the  $n$ -dark soliton mean-field solution, we now proceed to discuss the fluctuation part  $\hat{\phi}_n = \hat{\Psi} - \sqrt{N}\Phi_n e^{-i\mu_n t}$ . Following the standard Bogoliubov method [13], we write  $\hat{\phi}_n$  as,

$$\hat{\phi}_n(x, t) = \sum_j c_j^{(n)} u_j^{(n)}(x) e^{-i\omega_j t} + c_j^{(n)\dagger} v_j^{(n)*}(x) e^{i\omega_j t}. \quad (5)$$

Here  $c_j^{(n)\dagger}$  and  $c_j^{(n)}$  are quasi-particle creation and annihilation operators associated with the  $j$ -th collective mode. The mode functions  $u_j^{(n)}(x)$ ,  $v_j^{(n)}(x)$  and frequencies  $\omega_j$  are determined by

$$\mathcal{L} u_j^{(n)} + g \Phi_n^2 v_j^{(n)} = \omega_j u_j^{(n)} \quad (6)$$

$$\mathcal{L} v_j^{(n)} + g \Phi_n^{*2} u_j^{(n)} = -\omega_j v_j^{(n)} \quad (7)$$

where  $\mathcal{L} \equiv \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + 4g |\Phi_n|^2 - 2\mu_n \right)$ , under the restriction:  $\int dx (|u_j^{(n)}|^2 - |v_j^{(n)}|^2) = 1$ . Using the expansion given in (5), the Hamiltonian (1) is approximated by a diagonalized form:  $H \approx E_n + \sum_j \omega_j c_j^{(n)\dagger} c_j^{(n)}$ , where cubic and higher terms of  $\hat{\phi}_n$  are ignored.

Our task is to determine the incoherent part of atom density

$$\rho_n(x, t) \equiv \langle \hat{\phi}_n^\dagger(x, t) \hat{\phi}_n(x, t) \rangle \quad (8)$$

associated with the vacuum state in which there are no quasiparticles in all collective modes, i.e.,  $\langle c_j^{(n)\dagger} c_j^{(n)} \rangle = 0$ . It is not difficult to show that among all physical states, the vacuum gives the minimal time-averaged  $\rho_n(x, t)$ . With the help of Eq. (5),  $\rho_n(x, t)$  arising

from the vacuum fluctuations is given by:  $\rho_n(x) = \sum_j |v_j^{(n)}(x)|^2$ . An integration of  $\rho_n(x)$  over  $x$  gives the total number of incoherent atoms  $\delta N$ , i.e.,  $\delta N = \sum_j \delta N_j$ , with  $\delta N_j = \int dx |v_j^{(n)}(x)|^2$ .

We solve Eqs. (6) and (7) numerically for the first thousand collective modes in order to study the distribution of incoherent atoms. Although  $\delta N$  involves all collective modes, we find that only the low-frequency modes are significant. This is shown in Fig. 2 in which  $\delta N_j$  as a function of  $\omega_j$  is plotted. For the cases with soliton numbers up to four, we see that  $\delta N_j$  diminishes quickly as the frequency increases. If we superimpose the four cases in Fig. 2 together, the four tails associated with the high frequency part of  $\delta N_j$  all roughly lie on the same curve. This implies that the presence of more dark solitons essentially do not cause extra fluctuations in the high frequency modes.

From now on we will focus on low frequency modes. One prominent feature we discover is that *the number of modes with negative frequencies equals the number of dark solitons*. These particular modes are shown in Fig. 2 as black filled circles. In the Bogoliubov theory, collective modes with negative frequencies and positive norm (i.e.,  $\int dx (|u_j^{(n)}|^2 - |v_j^{(n)}|^2) = 1$ ) are known as ‘anomalous modes’. Previous studies on a single dark soliton [6] and a single vortex [14] have identified the particle-like motion associated with anomalous modes. For example, in the  $n = 1$  dark soliton case, the anomalous mode describes the small amplitude oscillations of the soliton at a frequency  $-\omega_T/\sqrt{2}$  in the Thomas-Fermi limit [3,5,6]. This agrees with the exact anomalous mode frequency shown in Fig. 2a within 1.3 percent. By increasing the interaction strength  $g$ , we find that the numerical frequency approaches  $-\omega_T/\sqrt{2}$ .

In Fig. 3 the incoherent atom density  $\rho_n(x)$  is plotted. We see that  $\rho_n(x)$  (solid lines) are strongly concentrated at the positions of the dark solitons. In each figure the contribution from the anomalous modes is shown by the dash line, which is the partial sum in Eq. (9), counting only the finite number of modes with  $\omega_j < 0$ . By comparing the curves, we conclude that *incoherent atoms near the dark solitons mainly originate from the vacuum fluctuations in the anomalous modes*. In fact, if we remove the background effect by defining the quantity  $\Delta(x) \equiv \rho_n(x) - \rho_0(x)$  as a measure of fluctuations due to the presence of dark solitons, we find a more striking evidence. This is shown in the inset of Fig. 3 in which the solid lines ( $\Delta(x)$ ) and the dash lines (from the anomalous modes) are almost indistinguishable from each other.

Having elucidated the significance of anomalous modes, we now make a connection to the particle-like motion of the dark solitons. The fact that the number of anomalous modes and the number of dark solitons are equal (see Fig. 2) provides an important clue. We can imagine the  $n$  mutually repulsive dark solitons, each being pushed toward the trap center by the inhomogeneous

background, form a one-dimensional ‘lattice’ at equilibrium. As a result, there are  $n$  normal modes of oscillations when the equilibrium system is slightly disturbed. These modes correspond to the anomalous modes that are well localized around the dark solitons. For example, the frequency  $\approx -\omega_T/\sqrt{2}$  appears in all four cases in Fig. 2. We find that this mode corresponds to the in-phase mode that the  $n$  solitons move together without changing their relative separations.

To elaborate the physical picture quantitatively, we present a method to estimate the effective masses and frequencies of the coupled oscillations of solitons. For definiteness, let us examine the  $n = 2$  case, and consider the ansatz wavefunction of the dark soliton pair,

$$\tilde{\Phi}_2(x, t) = \tilde{\alpha}\Phi_0(x)\psi_1(x, t)\psi_2(x, t). \quad (9)$$

Here  $\tilde{\alpha}$  is a normalization constant and  $\psi_j(x, t)$  is the exact solution of a dark soliton at the position  $q_j$  moving at a speed  $\dot{q}_j$  ( $j = 1, 2$ ) in a homogeneous background, i.e.,

$$\psi_j(x, t) = i\frac{\dot{q}_j}{v_s} + \sqrt{1 - \frac{\dot{q}_j^2}{v_s^2}} \tanh \left[ \sqrt{1 - \frac{\dot{q}_j^2}{v_s^2}} \left( \frac{x - q_j}{l_0} \right) \right], \quad (10)$$

where  $v_s = g^{1/2}\Phi_0(0)$  is the local speed of sound for the ground state at the origin. The construction of  $\tilde{\Phi}_2(x, t)$  generalizes the trial wavefunction (3) by including the speed  $\dot{q}_j$  as variational parameters. Our strategy is to determine how the energy of the system changes if  $q_j$  and  $\dot{q}_j$  slightly deviate from the stationary point:  $\dot{q}_j = 0$  and  $q_1 = -q_2 = Q$ . This can be done by inserting the ansatz (9) into Eq. (4) and keep only the quadratic terms. The energy change can be cast in the form of two harmonic oscillators,

$$\Delta E \approx \frac{1}{2}M(\dot{q}_+^2 + \dot{q}_-^2 + \omega_+^2 q_+^2 + \omega_-^2 q_-^2). \quad (11)$$

Here we have defined the generalized coordinates for the in-phase mode (+) and the out-of-phase mode (−) by:  $q_{\pm} = (\delta q_1 \pm \delta q_2)/\sqrt{2}$  with  $\delta q_1 = q_1 - Q$  and  $\delta q_2 = q_2 + Q$ .

We find that  $\omega_{\pm}$  obtained in Eq. (11) match quite well with the exact numerical values of the anomalous mode frequencies, noticing that ansatz (9) is not an exact solution. For example,  $\omega_+$  and  $\omega_-$  attain 93% and 96% accuracy respectively for the case with  $g = 200$ . The effective mass  $M$  for each soliton turns out to be negative, which is a basic property of dark solitons discussed in literature [4]. When  $g \gg 1$  is sufficiently large, we find  $M \approx -4N\Psi_0(0)^2 l_0 \approx -3.2Ng^{-2/3}$  (in units of the atom mass  $m$ ). Since  $g = N\bar{\eta}$  is proportional to  $N$ , the absolute value of the soliton mass  $M$  increases as  $N^{1/3}$  as the number of particles increases.

Treating the positions  $q_j$  and momenta  $p_j \equiv M\dot{q}_j$  as dynamical variables, we construct a *two-particle* Hamiltonian based on the energy (11). The quantized form of

the Hamiltonian obtained corresponds to the part of the Bogoliubov Hamiltonian containing the two anomalous modes. Therefore the creation and annihilation operators associated with the anomalous modes can be explicitly related to the position and momentum operators  $\hat{q}_j$  and  $\hat{p}_j$ . In this way the vacuum state of quasiparticles in the anomalous modes are now recognized as the zero-point oscillations of the coupled dark solitons [15].

For the case  $n = 2$  discussed here, the vacuum state gives the position fluctuations  $\langle \Delta q_{\pm}^2 \rangle = 1/2M\omega_{\pm}$ . Since the width of each dark solitons is about  $l_0$ , the fluctuations become significant near the notches if the ratio  $\langle \Delta q_j \rangle / l_0$  ( $j = 1, 2$ ) is greater or comparable to one. We find that this occurs if  $N^{1/6}\bar{\eta}^{2/3} > 1$ , according to the expression of  $M$  obtained above and the fact that  $\omega_{\pm}$  are order one. We can estimate the number density of incoherent atoms at the notches. For example, by counting only the contributions from the two anomalous modes, we have [16]

$$\hat{\phi}_2(x = \pm Q) \approx \sqrt{N}\tilde{\alpha}\frac{\Phi_0(0)}{\sqrt{2}} \left( i\frac{\hat{p}_+ \pm \hat{p}_-}{Mv_s} - \frac{\hat{q}_+ \pm \hat{q}_-}{l_0} \right). \quad (12)$$

Therefore the density  $\langle \hat{\phi}_2^\dagger(Q)\hat{\phi}_2(Q) \rangle$  is approximately  $3^{4/3}2^{-1/3}\tilde{\alpha}NM^{-1}g^{1/3}(\omega_+^{-1} + \omega_-^{-1})/32$ , which is an increasing function of  $g$ . Such an analytical estimation matches the exact numerical results in Fig. 3b with about 92% accuracy. If a higher accuracy is desired, the ansatz (9) should be generalized. The exact solution of a dark-soliton pair by Blow and Doran [12] in a homogeneous background should provide insights for making corrections.

To conclude, we have investigated the fundamental fluctuations in dark solitons arising from their Bogoliubov quasiparticle vacuum. These quantum fluctuations are revealed in the form of incoherent atoms strongly concentrated near the notches. We interpret such a quantum effect as a consequence of zero-point oscillations of the coupled dark solitons. By constructing a particle-like dynamical model with solitons’ positions and velocities as (finite numbers) degrees of freedom, quantitative predictions can be made approximately. In particular we find the regime  $N^{1/6}\bar{\eta}^{2/3} > 1$  where the position uncertainty becomes comparable with the soliton’s width. Therefore *blurred* dark solitons would occur if the particle number is high or the transverse area of the condensate is sufficiently small.

## ACKNOWLEDGMENTS

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  - [15] Dark solitons are excited states which do not exist at  $T = 0K$  in strict sense. We use the word ‘zero-point’ in order to emphasize the analogy with the intrinsic quantum uncertainties familiar in harmonic oscillators.
  - [16] To obtain Eq. (12), we made a Taylor expansion of  $\tilde{\Phi}_2$  in a power series of  $q_{\pm}$  and  $p_{\pm}$ . The linear terms approximate the part of  $\hat{\phi}_2$  with anomalous modes, after treating  $q_{\pm}$  and  $p_{\pm}$  as operators.

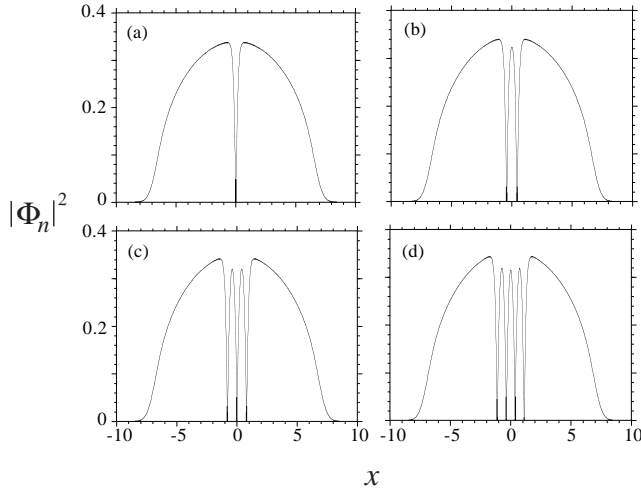


FIG. 1. The first four stationary dark soliton states obtained from the numerical solutions of Eq. (2) with  $g = 200$ : (a)  $n = 1$ , (b)  $n = 2$ , (c)  $n = 3$  and (d)  $n = 4$ . The spatial coordinate  $x$  is in units of  $\sqrt{\hbar/m\omega_T}$ .

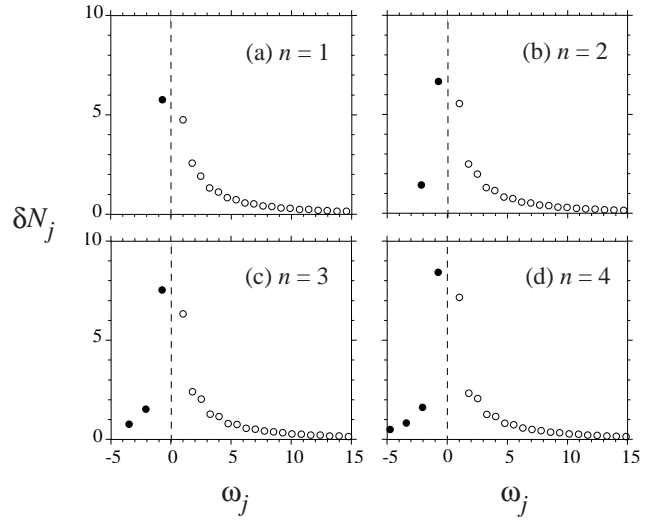


FIG. 2. The number of incoherent atoms associated with each collective excitation mode with the frequency  $\omega_j$ . The four plots correspond to the first four dark soliton states in Fig. 1 with the same interaction strength  $g = 200$ . The negative frequencies modes are anomalous modes plotted in black solid circles. The frequencies  $\omega_j$  are in units of  $\omega_T$ .

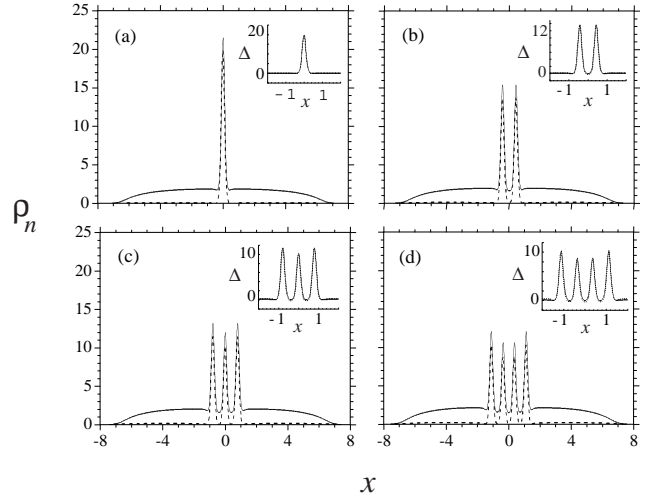


FIG. 3. Spatial density profiles of incoherent atoms (solid lines): (a)  $n = 1$ , (b)  $n = 2$ , (c)  $n = 3$  and (d)  $n = 4$  with the same parameters used in Fig. 1. The dash line in each figure indicates the contribution from the anomalous modes only. The solid line in each inset shows the quantity  $\Delta = \rho_n(x) - \rho_0(x)$  as a function of  $x$ . In each inset we have also plotted the contribution from the anomalous modes in a dash line (that appears almost exactly the same as  $\Delta$ ).